Exact N-soliton solution of the modified nonlinear Schrödinger equation

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(Received 26 January 1993)

By use of Hirota's direct method, exact N-soliton solutions have been obtained for the modified nonlinear Schrödinger equation.

PACS number(s): 42.81.Dp, 42.65.−k, 42.50.Rh, 52.35.Sb

I. INTRODUCTION

The modulated Alfvén wave propagating along a magnetic field in cold plasmas is described by the modified nonlinear Schrödinger (MNLS) equation [1]

$$i\frac{\partial q}{\partial z}+\frac{\alpha}{2}\frac{\partial^2 q}{\partial t^2}+\beta|q|^2q+i\gamma\frac{\partial(|q|^2q)}{\partial t}=0 , \quad (1)$$

where α, β, γ are real parameters, β is inverse proportional to the wavelength. For the long wavelength, Eq. (1) reduces to the derivative nonlinear Schrödinger (DNLS) equation [2]. In addition, the femtosecond-optical-pulse propagation in a monomodal optical fiber, where the characteristic length of the envelope is the same order as the wavelength of the carrier wave, is described by Eq. (1) [3]. For a shorter wavelength or a longer characteristic length, Eq. (1) reduces to the nonlinear Schrödinger (NLS) equation [4]. The MNLS equation has been solved by the method of meromorphic matrix of transformation [5,6]. However, it is a long and tedious process. Hirota's direct method [7–11], by which most of the exact explicit soliton solutions of the nonlinear partial differential equation had been given, seems to be unsuitable.

The purpose of the present paper is to derive an exact explicit N-soliton solution by use of Hirota's direct method.

II. EXACT N-SOLITON SOLUTIONS

Exact N-soliton solutions of Eq. (1) can be expressed in the form

$$q(z,t)=\frac{g(z,t)[f(z,t)]^*}{[f(z,t)]^2} , \quad (2)$$

where

$$f(z,t)=\sum_{\mu=0,1}{}' \exp\left[\sum_{\substack{i,j\\(i<j)}}^{(2N)}\varphi_{(i,j)}\mu_i\mu_j+\sum_{i=1}^{N}\mu_i\varphi_i+\sum_{i=1}^{2N}\mu_i\eta_i\right] , \quad (3)$$

$$f^*(z,t)=\sum_{\mu=0,1}{}' \exp\left[\sum_{\substack{i,j\\(i<j)}}^{(2N)}\varphi_{(i,j)}\mu_i\mu_j+\sum_{i=N+1}^{2N}\mu_i\varphi_i+\sum_{i=1}^{2N}\mu_i\eta_i\right] , \quad (4)$$

$$g(z,t)=\sum_{\nu=0,1}{}'' \exp\left[\sum_{\substack{i,j\\(i<j)}}^{(2N)}\varphi_{(i,j)}\nu_i\nu_j+\sum_{i=N+1}^{2N}\nu_i\varphi_i+\sum_{i=1}^{2N}\nu_i\eta_i\right] , \quad (5)$$

$$g^*(z,t)=\sum_{\nu=0,1}{}''' \exp\left[\sum_{\substack{i,j\\(i<j)}}^{2N}\varphi_{(i,j)}\nu_i\nu_j+\sum_{i=1}^{N}\nu_i\varphi_i+\sum_{i=1}^{2N}\nu_i\eta_i\right] , \quad (6)$$

where

$$\eta_i=K_iz+\Omega_it+\eta_i^0,\quad K_i=i\frac{\alpha}{2}\Omega_i^2,\quad \varphi_i=\ln\frac{(\beta+i\gamma\Omega_i)}{\alpha} ,$$
$$\eta_{i+N}=\eta_i^*,\quad \Omega_{i+N}=\Omega_i^*,\quad \varphi_{i+N}=\varphi_i^* \quad \text{for } i=1,2,\ldots,N , \quad (7)$$

where $*$ implies a complex conjugate, and

$$\varphi_{(i,j)}=\begin{cases}-2\ln(\Omega_i+\Omega_j) & \text{for } i=1,2,\ldots,N \text{ and } j=N+1,N+2,\ldots,2N , \quad (8)\\ 2\ln(\Omega_i-\Omega_j) & \text{for } i=1,2,\ldots,N \text{ and } j=1,2,\ldots,N \\ & \text{or } i=N+1,N+2,\ldots,2N \text{ and } j=N+1,N+2,\ldots,2N , \quad (9)\end{cases}$$

where $\Omega_i=\tau_i-i\omega_i$ and η_i^0 are complex parameters of the ith soliton, $\sum'_{\mu=0,1}$ indicates the summation over all possible combinations of $\mu_1=0,1$, $\mu_2=0,1,\ldots,$ $\mu_{2N}=0,1$, under the condition $\sum_{i=1}^{N}\mu_i=\sum_{i=1}^{N}\mu_{i+N}$, $\sum''_{\nu=0,1}$ and $\sum'''_{\nu=0,1}$ indicate the summations over all possible combinations of $\nu_1=0,1$, $\nu_2=0,1,\ldots,$ $\nu_{2N}=0,1$ under the conditions

$\sum_{i=1}^{N}\nu_i=1+\sum_{i=1}^{N}\nu_{i+N}$ and $1+\sum_{i=1}^{N}\nu_i=\sum_{i=1}^{N}\nu_{i+N}$, respectively, and $\sum_{i,j\,(i<j)}^{(2N)}$ indicates the summation over all possible pairs taken from $2N$ elements with the specified condition $i<j$, as indicated. We assume all Ω_i are different from each other.

As an example, we write forms of f and g for $N=2$

$$\begin{aligned} f(z,t)=1&+\frac{\beta+i\gamma\Omega_1}{\alpha(\Omega_1+\Omega_1^*)^2}\exp(\eta_1+\eta_1^*)+\frac{\beta+i\gamma\Omega_1}{\alpha(\Omega_1+\Omega_2^*)^2}\exp(\eta_1+\eta_2^*)\\ &+\frac{\beta+i\gamma\Omega_2}{\alpha(\Omega_2+\Omega_1^*)^2}\exp(\eta_2+\eta_1^*)+\frac{\beta+i\gamma\Omega_2}{\alpha(\Omega_2+\Omega_2^*)^2}\exp(\eta_2+\eta_2^*)\\ &+\frac{(\Omega_1-\Omega_2)^2(\Omega_1^*-\Omega_2^*)^2(\beta+i\gamma\Omega_1)(\beta+i\gamma\Omega_2)}{\alpha^2(\Omega_1+\Omega_1^*)^2(\Omega_1+\Omega_2^*)^2(\Omega_2+\Omega_2^*)^2(\Omega_2+\Omega_1^*)^2}\exp(\eta_1+\eta_2+\eta_1^*+\eta_2^*)\,, \end{aligned} \tag{10}$$

$$\begin{aligned} g(z,t)=\exp(\eta_1)+\exp(\eta_2)&+\frac{(\Omega_1-\Omega_2)^2(\beta-i\gamma\Omega_1^*)}{\alpha(\Omega_1+\Omega_1^*)^2(\Omega_2+\Omega_1^*)^2}\exp(\eta_1+\eta_2+\eta_1^*)\\ &+\frac{(\Omega_1-\Omega_2)^2(\beta-i\gamma\Omega_2^*)}{\alpha(\Omega_1+\Omega_2^*)^2(\Omega_2+\Omega_2^*)^2}\exp(\eta_1+\eta_2+\eta_2^*)\,. \end{aligned} \tag{11}$$

Substituting Eq. (2) into Eq. (1), we can obtain

$$\left[iD_z+\frac{\alpha}{2}D_t^2\right](gf)=0\,, \tag{12}$$

$$\left[gf\left[iD_z+\frac{\alpha}{2}D_t^2\right]+\alpha D_t(gf)D_t\right](f^*f)-\alpha gf^*D_t^2(ff)+[(\beta+i3\gamma D_t)(gf)+i\gamma gfD_t](g^*g)=0\,, \tag{13}$$

where

$$D_z^mD_t^n(gf)=\left[\frac{\partial}{\partial z}-\frac{\partial}{\partial z'}\right]^m\left[\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right]^n g(z,t)f(z',t')|_{\text{at } z'=z,t'=t}\,. \tag{14}$$

It is evident that $q(z,t)$, defined by Eqs. (2)–(9), is a solution of Eq. (1) provided that f and g satisfy Eqs. (12) and (13).

Substituting the expressions for f and g into Eqs. (12) and (13), we have

$$\sum_{\nu=0,1}''\sum_{\mu=0,1}'\left[i\sum_{i=1}^{2N}K_i(\nu_i-\mu_i)+\frac{\alpha}{2}\left[\sum_{i=1}^{2N}\Omega_i(\nu_i-\mu_i)\right]^2\right]\times\exp\left[\sum_{\substack{i,j\\(i<j)}}^{(2N)}\varphi_{(i,j)}(\nu_i\nu_j+\mu_i\mu_j)+\sum_{i=1}^{N}(\mu_i\varphi_i+\nu_{i+N}\varphi_{i+N})+\sum_{i=1}^{2N}(\nu_i+\mu_i)\eta_i\right]=0 \tag{15}$$

and

$$\begin{aligned} \sum_{\nu=0,1}''\sum_{\mu=0,1}'\Bigg\{&\sum_{\mu'=0,1}'\sum_{\mu''=0,1}'\left[i\sum_{i=1}^{2N}K_i(\mu_i'-\mu_i'')+\frac{\alpha}{2}\left[\sum_{i=1}^{2N}\Omega_i(\mu_i'-\mu_i''\right]^2\right.\\ &\left.+\alpha\sum_{i=1}^{2N}\Omega_i(\nu_i-\mu_i)\sum_{i=1}^{2N}\Omega_i(\mu_i'-\mu_i'')-\alpha\left[\sum_{i=1}^{2N}\Omega_i(\mu_i-\mu_i'')\right]^2\right]\\ &\times\exp\left[\sum_{\substack{i,j\\(i<j)}}^{(2N)}\varphi_{(i,j)}(\mu_i'\mu_j'+\mu_i''\mu_j'')+\sum_{i=1}^{N}(\mu_i''\varphi_i+\mu_{i+N}'\varphi_{i+N})+\sum_{i=1}^{2N}(\mu_i'+\mu_i'')\eta_i\right]\\ &+\sum_{\nu'=0,1}'''\sum_{\nu''=0,1}''\left[\beta+i3\gamma\sum_{i=1}^{2N}\Omega_i(\nu_i-\mu_i)+i\gamma\sum_{i=1}^{2N}\Omega_i(\nu_i'-\nu_i'')\right]\\ &\times\exp\left[\sum_{\substack{i,j\\(i<j)}}^{(2N)}\varphi_{(i,j)}(\nu_i'\nu_j'+\nu_i''\nu_j'')+\sum_{i=1}^{N}(\nu_i'\varphi_i+\nu_{i+N}''\varphi_{i+N})+\sum_{i=1}^{2N}(\nu_i'+\nu_i'')\eta_i\right]\Bigg\}\\ &\times\exp\left[\sum_{\substack{i,j\\(i<j)}}^{(2N)}\varphi_{(i,j)}(\nu_i\nu_j+\mu_i\mu_j)+\sum_{i=1}^{N}(\mu_i\varphi_i+\nu_{i+N}\varphi_{i+N})+\sum_{i=1}^{2N}(\nu_i+\mu_i)\eta_i\right]=0\,. \end{aligned} \tag{16}$$

In Eq. (15), the coefficient of the factor

$$\exp\left[\sum_{i=1}^{L}\eta_i+\sum_{i=1}^{L'}\eta_{i+N}+\sum_{i=L+1}^{L+M}2\eta_i+\sum_{i=L'+1}^{L'+M'}2\eta_{i+N}\right]$$

is

$$D_1=\sum_{\nu=0,1}{}''\sum_{\mu=0,1}{}''\,\mathscr{C}_{\text{condition}}(\nu,\mu)\left[i\sum_{i=1}^{2N}K_i(\nu_i-\mu_i)+\frac{\alpha}{2}\left[\sum_{i=1}^{2N}\Omega_i(\nu_i-\mu_i)\right]^2\right]\times\exp\left[\sum_{\substack{i,j\\(i<j)}}^{(2N)}\varphi_{(i,j)}(\nu_i\nu_j+\mu_i\mu_j)+\sum_{i=1}^{N}(\mu_i\varphi_i+\nu_{i+N}\varphi_{i+N})\right], \tag{17}$$

where $\mathscr{C}_{\text{condition}}(\nu,\mu)$ implies that the summations over ν and μ should be performed under the following conditions:

$\nu_i+\mu_i=1$ for $i=1,2,\ldots,L$ or $i-N=1,2,\ldots,L'$,

$\nu_i=\mu_i=1$ for $i=L+1,L+2,\ldots,L+M$ or $i-N=L'+1,L'+2,\ldots,L'+M'$,

$\nu_i=\mu_i=0$ for $i=L+M+1,L+M+2,\ldots,N$ or $i-N=L'+M'+1,L'+M'+2,\ldots,N$.

Under the above conditions, we find that the conditions of ν and μ in Eq. (15),

$$\sum_{i=1}^{N}\mu_i=\sum_{i=1}^{N}\mu_{i+N}\quad\text{and}\quad\sum_{i=1}^{N}\nu_i=1+\sum_{i=1}^{N}\nu_{i+N},$$

are compatible and mutually convertible from one to the other if, and only if

$$L+2M=1+L'+2M'.$$

Let $\sigma_i=\nu_i-\mu_i$ for $i=1,2,\ldots,2N$ under the same conditions. We have

$$i\sum_{i=1}^{2N}K_i(\nu_i-\mu_i)+\frac{\alpha}{2}\left[\sum_{i=1}^{2N}\Omega_i(\nu_i-\mu_i)\right]^2=i\left[\sum_{i=1}^{L}+\sum_{i=N+1}^{N+L'}\right]K_i\sigma_i+\frac{\alpha}{2}\left[\left[\sum_{i=1}^{L}+\sum_{i=N+1}^{N+L'}\right]\Omega_i\sigma_i\right]^2 \tag{18}$$

and

$$\sum_{\substack{i,j\\(i<j)}}^{(2N)}\varphi_{(i,j)}(\mu_i\mu_j+\nu_i\nu_j)+\sum_{i=1}^{N}(\mu_i\varphi_i+\nu_{i+N}\varphi_{i+N})=\sum_{\substack{i,j\\(j>i=1)}}^{(L)}\varphi_{(i,j)}\frac{1+\sigma_i\sigma_j}{2}+\sum_{\substack{i,j\\(j>i=N+1)}}^{(N+L')}\varphi_{(i,j)}\frac{1+\sigma_i\sigma_j}{2}+\sum_{i=1}^{L}\sum_{j=N+1}^{N+L'}\varphi_{(i,j)}\frac{1+\sigma_i\sigma_j}{2}+\sum_{i=1}^{L}\frac{1-\sigma_i}{2}\varphi_i+\sum_{i=N+1}^{N+L'}\frac{1+\sigma_i}{2}\varphi_i+\text{const (independent of } \sigma). \tag{19}$$

The condition of μ in Eq. (17),

$$\sum_{i=1}^{N}\mu_i=\sum_{i=1}^{N}\mu_{i+N},$$

is converted to

$$\sum_{i=1}^{L}\sigma_i-\sum_{i=N+1}^{N+L'}\sigma_i=1. \tag{20}$$

Hence, we have

$$D_1=\text{const}\times\hat{D}_1(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})=\text{const}\times\sum_{\sigma=\pm1}h(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})b(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'}), \tag{21}$$

where

$$h(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})=i\left[\sum_{i=1}^{L}+\sum_{i=N+1}^{N+L'}\right]K_i\sigma_i+\frac{\alpha}{2}\left[\left[\sum_{i=1}^{L}+\sum_{i=N+1}^{N+L'}\right]\Omega_i\sigma_i\right]^2, \tag{22}$$

$$\begin{aligned}b(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,Q_{N+L'})=&\prod_{i=1}^{L}\left[\frac{\beta+i\gamma\Omega_i}{\alpha}\right]^{(1-\sigma_i)/2}\\&\times\prod_{i=N+1}^{N+L'}\left[\frac{\beta-i\gamma\Omega_i}{\alpha}\right]^{(1+\sigma_i)/2}\times\prod_{\substack{i,j\\(j>i=1)}}^{(L)}(\Omega_i-\Omega_j)^{1+\sigma_i\sigma_j}\\&\times\prod_{\substack{i,j\\(j>i=N+1)}}^{(N+L')}(\Omega_i-\Omega_j)^{1+\sigma_i\sigma_j}\\&\times\prod_{i=1}^{L}\prod_{j=N+1}^{N+L'}(\Omega_i+\Omega_j)^{-(1+\sigma_i\sigma_j)},\end{aligned} \tag{23}$$

where $\sum_{\sigma=\pm1}$ implies the summation over all possible combinations of $\sigma_1=\pm1,\sigma_2=\pm1,\ldots,\sigma_L,\sigma_{N+1},\ldots,\sigma_{N+L'}=\pm1$ under the condition Eq. (20) and $\prod_{i,j\,(j>i=1)}^{(L)}$ indicates the product of all possible combinations of pairs chosen from L elements with the specified condition $j>i$.

Similar procedures give, for the coefficient of the factor

$$\exp\left[\sum_{i=1}^{l_1}\eta_i+\sum_{i=N+1}^{l_1'}\eta_i+\sum_{i=l_1+1}^{l_1+l_2}2\eta_i+\sum_{i=N+l_1'+1}^{N+l_1'+l_2'}2\eta_i+\sum_{i=l_1+l_2+1}^{L}3\eta_i+\sum_{i=N+l_1'+l_2'+1}^{N+L'}3\eta_i+\sum_{i=L+1}^{L+M}4\eta_i+\sum_{i=N+L'+1}^{N+L'+M'}4\eta_i\right]$$

in Eq. (16), that

$$\begin{aligned}D_2&=\text{const}\times\hat{D}_2(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})\\&=\text{const}\times\sum_{\sigma}\left[\sum_{\sigma'}h_1(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})b_1(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})\right.\\&\qquad\left.+\sum_{\sigma''}h_2(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})b_1(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})\right],\end{aligned} \tag{24}$$

where

$$\begin{aligned}h_1(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})=&\,i\left[\sum_{i=1}^{L}+\sum_{i=N+1}^{N+L'}\right]K_i\sigma_i'+\frac{\alpha}{2}\left[\left[\sum_{i=1}^{L}+\sum_{i=N+1}^{N+L'}\right]\Omega_i\sigma'\right]^2\\&+\alpha\left[\sum_{i=1}^{L}+\sum_{i=N+1}^{N+L'}\right]\Omega_i\sigma_i\left[\sum_{i=1}^{L}+\sum_{i=N+1}^{N+L'}\right]\Omega_i\sigma_i'\\&-\alpha\left[\left[\sum_{i=1}^{L}+\sum_{i=N+1}^{N+L'}\right]\Omega_1\frac{\delta_i-\sigma_i-\delta_i'+\sigma_i'}{2}\right]^2,\end{aligned} \tag{25}$$

$$h_2(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})=\beta+i3\gamma\left[\sum_{i=1}^{L}+\sum_{i=N+1}^{N+L'}\right]\Omega_i\sigma_i+i\gamma\left[\sum_{i=1}^{L}+\sum_{i=N+1}^{N+L'}\right]\Omega_i\sigma_i'', \tag{26}$$

and

$$\begin{aligned}&b_1(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})\\&\quad=\prod_{i=1}^{L}\left[\frac{\beta+i\gamma\Omega_i}{\alpha}\right]^{(\delta_i-\sigma_i+\delta_i'-\sigma_i')/2}\\&\qquad\times\prod_{i=N+1}^{N+L'}\left[\frac{\beta-i\gamma\Omega_i}{\alpha}\right]^{(\delta_i+\sigma_i+\delta_i'+\sigma_i')/2}\\&\qquad\qquad\times\left[\prod_{\substack{i,j\\(j>i=1)}}^{(L)}(\Omega_i-\Omega_j)\prod_{\substack{i,j\\(j>i=N+1)}}^{(N+L')}(\Omega_i-\Omega_j)\prod_{i=1}^{L}\prod_{j=N+1}^{N+L'}(\Omega_i+\Omega_j)^{-1}\right]^{\delta_i\delta_j+\sigma_i\sigma_j+\delta_i'\delta_j'+\sigma_i'\sigma_j'},\end{aligned} \tag{27}$$

with

$$\delta_i=\nu_i+\mu_i,\quad \delta'_i=\mu'_i+\mu''_i,\quad \delta''_i=\nu'_i+\nu''_i\ ,$$

$$\sigma_i=\nu_i-\mu_i,\quad \sigma_i=\mu'_i-\mu''_i,\quad \sigma_i=\nu'_i-\nu''_i$$

under the following conditions:

$\delta_i+\delta'_i=1$ and $\delta_i+\delta''_i=1$
for $i=1,2,\ldots,l_1$ or $i-N=1,2,\ldots,l'_1$.

$\delta_i+\delta'_i=2$ and $\delta_i+\delta''_i=2$
for $i=l_1+1,l_1+2,\ldots,l_1+l_2$
or $i-N=l'_1+1,l'_1+2,\ldots,l'_1+l'_2$,

$\delta_i+\delta'_1=3$ and $\delta_i+\delta''_i=3$
for $i=l_1+l_2+1,l_1+l_2+2,\ldots,L$
or $i-N=l'_1+l'_2+1,l'_1+l'_2+2,\ldots,L'$,

$\delta_i=\delta'_i=\delta''_i=2$ for $i=L+1,L+2,\ldots,L+M$
or $i-N=L'+1,L'+2,\ldots,L'+M'$,

$\delta_i=\delta'_i=\delta''_i=0$ for $i=L+M+1,L+M+2,\ldots,N$
or $i-N=L'+M'+1,L'+M'+2,\ldots,N$,

where $\sum_\sigma$, $\sum_{\sigma'}$, and $\sum_{\sigma''}$ imply the summation over all possible combinations of σ_i, σ'_i, and $\sigma''_i=(0,\pm1)$ for $i=1,2,\ldots,L,N+1,\ldots,N+L'$ under the condition $\sum_{i=1}^{L}\sigma_i=1+\sum_{i=N+1}^{N+L'}\sigma_i, \sum_{i=1}^{L}\sigma'_i=\sum_{i=N+1}^{N+L'}\sigma'_i$ and $\sum_{i=1}^{L}\sigma''_i=-2+\sum_{i=N+1}^{N+L'}\sigma''_i$. Thus f and g are solutions of Eqs. (12) and (13) provided that the following identities hold:

$$\hat{D}_1(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})=0 \quad \text{for odd } n=L+L' \tag{28}$$

and

$$\hat{D}_2(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'}=0 \quad \text{for even } n=L+L'\ . \tag{29}$$

We shall prove the identities by mathematical induction. The identities $\hat{D}_1=0$ and $\hat{D}_2=0$ are easily verified for $n=1$ and 2, respectively. Now, assume that the identities hold for $n=L+L'-2$. From Eqs. (21) and (24), we can obtain [10]

$$\hat{D}_1(\Omega_3,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})=C_1^{L+L'-2}\prod_{\substack{i,j\\(j>i=3)}}^{(L)}(\Omega_i-\Omega_j)^2\prod_{\substack{i,j\\(j>i=N+1)}}^{N+L'}(\Omega_i-\Omega_j)^2\prod_{i=3}^{L}\prod_{j=N+1}^{N+L'}(\Omega_i+\Omega_j)^{-2}\ , \tag{30}$$

$$\begin{aligned}\hat{D}_2(\Omega_3,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})\\ =C_2^{L+L'-2}&\prod_{i=3}^{l_1}\prod_{j(>i)=4}^{L}(\Omega_i-\Omega_j)^2\prod_{i=N+1}^{N+l'_1}\prod_{j(>i)=N+2}^{N+L'}(\Omega_i-\Omega_j)^2\prod_{i=3}^{l_1}\prod_{j=N+1}^{N+L'}(\Omega_i+\Omega_j)^{-2}\prod_{i=l_1+1}^{l_1+l_2}\prod_{j(>i)=l_1+2}^{L}(\Omega_i-\Omega_j)^4\\ &\times\prod_{i=N+l'_1+1}^{N+l'_1+l'_2}\prod_{j(>i)=N+l'_1+2}^{N+L'}(\Omega_i-\Omega_j)^4\prod_{i=l_1+1}^{l_1+l_2}\prod_{j=N+1}^{N+L'}(\Omega_i+\Omega_j)^{-4}\\ &\times\prod_{\substack{i,j\\(j>i=l_i+l_2+1)}}^{(L)}(\Omega_i-\Omega_j)^6\prod_{\substack{i,j\\(j>i=N+l'_1+l'_2+1)}}^{N+L'}(\Omega_i-\Omega_j)^6\prod_{i=l_1+l_2+1}^{L}\prod_{j=N+1}^{N+L'}(\Omega_i+\Omega_j)^{-6}\ ,\end{aligned} \tag{31}$$

$$\hat{D}_1(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})=C_1^{L+L'}\prod_{\substack{i,j\\(j>i=1)}}^{(L)}(\Omega_i-\Omega_j)^2\prod_{\substack{i,j\\(j>i=N+1)}}^{(N+L')}(\Omega_i-\Omega_j)^2\prod_{i=1}^{L}\prod_{j=N+1}^{N+L'}(\Omega_i+\Omega_j)^{-2}\ , \tag{32}$$

and

$$\begin{aligned}\hat{D}_2(\Omega_1,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'})\\ =C_2^{L+L'}&\prod_{i=1}^{l_1}\prod_{j(>i)=2}^{L}(\Omega_i-\Omega_j)^2\prod_{i=N+1}^{N+l'_1}\prod_{j(>i)=N+2}^{N+L'}(\Omega_i-\Omega_j)^2\prod_{i=1}^{l_1}\prod_{j=N+1}^{N+L'}(\Omega_i+\Omega_j)^{-2}\\ &\times\prod_{i=l_1+1}^{l_1+l_2}\prod_{j(>i)=l_1+2}^{L}(\Omega_i-\Omega_j)^4\prod_{i=N+l'_1+1}^{N+l'_1+l'_2}\prod_{j(>i)=N+l'_1+2}^{N+L'}(\Omega_i-\Omega_j)^4\prod_{i=l_1+1}^{l_1+l_2}\prod_{j=N+1}^{N+L'}(\Omega_i+\Omega_j)^{-4}\\ &\times\prod_{\substack{i,j\\j>i=l_1+l_2+1}}^{(L)}(\Omega_i-\Omega_j)^6\prod_{\substack{i,j\\j>i=N+l'_1+l'_2+1}}^{(N+L')}(\Omega_i-\Omega_j)^6\prod_{i=l_1+l_2+1}^{L}\prod_{j=N+1}^{N+L'}(\Omega_i+\Omega_j)^{-6}\ ,\end{aligned} \tag{33}$$

and $C_1^{L+L'}$ and $C_2^{L+L'}$ can be expressed as

$$C_1^{L+L'}=\left[\sum_{i=1}^{L}+\sum_{i=N+1}^{N+L'}\right]a_i\Omega_i+\left[\left[\sum_{i=1}^{L}+\sum_{i+N+1}^{N+L'}\right]b_i\Omega_i\right]^2 \tag{34}$$

and

$$C_2^{L+L'}=\left[\sum_{i=1}^{L}+\sum_{i=N+1}^{N+L'}\right]a_i'\Omega_i+\left[\left[\sum_{i=1}^{L}+\sum_{i+N+1}^{N+L'}\right]b_i'\Omega_i\right]^2 \tag{35}$$

because $\Omega_i=\Omega_j=0$ for $\Omega_i\neq\Omega_j$ chosen from $\Omega_i,\Omega_2,\ldots,\Omega_L,\Omega_{N+1},\ldots,\Omega_{N+L'},C_1^{L+L'}$ and $C_2^{L+L'}$ become $C_1^{L+L'-2}=0$ and $C_2^{L+L''-2}=0$. Thus $\hat{D}_1$ and $\hat{D}_2$ must be zero for $n=L+L'$, and the identities have been proved.

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